Problem 1. (12 points)

(a) State the formula for the n-th Taylor polynomial of the function $f(x)$ at the number c.

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \quad , \quad \text{or}$$

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

(b) Find the third Taylor polynomial of the function $f(x) = \sqrt[4]{x}$ at 1.

$$f'(x) = \frac{1}{4}x^{-3/4}, \quad f''(x) = -\frac{3}{16}x^{-7/4}, \quad f'''(x) = \frac{21}{64}x^{-11/4}.$$

$$f'(1) = \frac{1}{4}, \quad f''(1) = -\frac{3}{16}, \quad f'''(1) = \frac{21}{64}.$$

$$P_3(x) = 1 + \frac{1}{4}(x-1) - \frac{3}{16}\cdot\frac{1}{2}(x-1)^2 + \frac{21}{64}\cdot\frac{1}{3!}(x-1)^3$$

(c) Use the polynomial from part (b) to estimate $\sqrt[4]{1.3}$. You do not need to simplify your answer.

$$\sqrt[4]{1.3} \approx P_3(1.3) =$$

$$1 + \frac{1}{4}(0.3) - \frac{3}{32}(0.3)^2 + \frac{21}{64\cdot 6}(0.3)^3$$

(d) Use Lagrange's Theorem to find an upper bound for the error in the above estimate.

$$f^{(4)}(x) = -\frac{11\cdot 21}{4\cdot 64}x^{-15/4}, \quad \text{so} \quad |f^{(4)}(x)| = \frac{11\cdot 21}{4\cdot 64\, x^{15/4}}.$$

This function is decreasing, so its max. on $[1, 1.3]$ is $\frac{11\cdot 21}{4\cdot 64}$.

Therefore,

$$\left|\sqrt[4]{1.3} - P_3(1.3)\right| < \frac{11\cdot 21}{4\cdot 64}\,\frac{(0.3)^4}{4!}.$$

Problem 2. (6 points)

(a) Find the first four terms of the Taylor series for the function $\ln(1+\sqrt{x})$ at 0.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \text{ so}$$

$$\ln(1+\sqrt{x}) = \sqrt{x} - \frac{x}{2} + \frac{x^{3/2}}{3} - \frac{x^2}{4} + \cdots$$

(b) Use the series from part (a) to estimate $\int_0^1 \ln(1+\sqrt{x})dx$. You do not need to simplify your answer.

$$\int_0^1 \ln(1+\sqrt{x})\,dx \approx \int_0^1 \left(\sqrt{x} - \frac{x}{2} + \frac{x^{3/2}}{3} - \frac{x^2}{4}\right) dx =$$

$$\left(\frac{2x^{3/2}}{3} - \frac{1}{2}\cdot\frac{x^2}{2} + \frac{1}{3}\cdot\frac{2}{5}x^{5/2} - \frac{1}{4}\cdot\frac{x^3}{3}\right)\Big|_0^1 =$$

$$\frac{2}{3} - \frac{1}{4} + \frac{2}{15} - \frac{1}{12}$$

Problem 3. (6 points) Determine whether the following sequences converge. In case of convergence, find the limit. In case of divergence, explain why the sequence diverges.

(a) $a_n = \dfrac{3+5n}{1+n^2}$

$$\lim_{n\to\infty} \frac{3+5n}{1+n^2} = \lim_{n\to\infty} \frac{\frac{3}{n^2}+\frac{5}{n}}{\frac{1}{n^2}+1} = \frac{0}{1} = 0.$$

(b) $a_1 = 1,\ a_{n+1} = 1 + 2a_n$

$a_2 = 1+2(1) = 3$, $a_3 = 1+2(3) = 7$,

$a_4 = 1+2(7) = 15$, $a_5 = 1+2(15) = 31$.

In general, $a_n = 2^n - 1$. Clearly a_n is unbounded, so it diverges.

Problem 4. (8 points) Determine whether the following series converge. In case of convergence, find the value of the sum. In case of divergence, explain why the series diverges.

(a) $\sum_{n=1}^{\infty} 2^{2n} \cdot 3^{1-n}$ $= \sum_{n=1}^{\infty} 2^{2n} \cdot 3^{-n} \cdot 3 = 3 \cdot \sum_{n=1}^{\infty} 2^{2n}/3^n =$

$3 \cdot \sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$. Since this is a geometric series with $|4/3| > 1$, the series diverges.

(b) $\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+2)}\right)$ $= \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n + \sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

Since $|1/e| < 1$, $\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n = \frac{1}{1-1/e} - 1$.

Now, $\frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$, so

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{2} \cdot \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{1}{2}\left(1 - \cancel{\tfrac{1}{3}} + \frac{1}{2} - \cancel{\tfrac{1}{4}} + \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}} + \cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{6}} + \ldots\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}.$$

Therefore $\sum_{n=1}^{\infty} \frac{1}{e^n} + \frac{1}{n(n+2)} = \frac{1}{1 - \frac{1}{e}} - 1 + \frac{3}{4}$.

Problem 5. (BONUS problem – 3 points)

Let s_n be the n-th partial sum of the harmonic series. We proved in class that $s_n > \ln n$ by comparing s_n to an integral of the function $1/x$. Let t_n be the sequence defined by the formula $t_n = s_n - \ln(n)$. Show that t_n is decreasing and bounded below, and therefore convergent. (Hint: interpret $t_n - t_{n+1}$ as a difference of areas.)

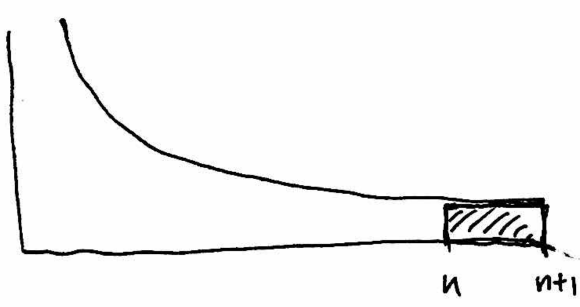
Since $t_n > 0$, in order to show that t_n converges it is enough to show that t_n is decreasing.

Note that
$$t_n - t_{n+1} = 1+\frac{1}{2}+\dots+\frac{1}{n} - \ln(n) - \left(1+\frac{1}{2}+\dots+\frac{1}{n+1} - \ln(n+1)\right)$$
$$= \ln(n+1) - \ln(n) - \frac{1}{n+1}.$$

To show that t_n is decreasing we need to show that
$$\ln(n+1) - \ln(n) > \frac{1}{n+1}.$$

To see this, note that $\ln(n+1) = \int_1^{n+1} \frac{dx}{x}$ and $\ln(n) = \int_1^n \frac{dx}{x}$, so $\ln(n+1) - \ln(n) = \int_n^{n+1} \frac{dx}{x}$.

Thus, $\ln(n+1) - \ln(n)$ represents the area under the curve $y = \frac{1}{x}$ from n to $n+1$. This area is greater than the area of the shaded rectangle shown here, which is $\frac{1}{n+1}$.



This shows that $\ln(n+1) - \ln(n) > \frac{1}{n+1}$, so t_n is decreasing and therefore convergent.

FYI: The limit of t_n is ≈ 0.577. This is called the Euler constant γ.